Q No.→ Show that l_2 is a Hilbert Space with respect to the inner Product defined by

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i} \text{ for all sequences } x = (x_i), y = (y_i)$$

of l_2.

Soln: Since $x, y \in l_2$, the series $\sum |x_i|^2$ & $\sum |y_i|^2$ — ① are Convergent.

For each +ve integer n,

$(|x_i| - |y_i|)^2 \geq 0$

So that $2|x_i y_i| \leq |x_i|^2 + |y_i|^2$ —— ②

Now, $\sum (|x_i|^2 + |y_i|^2)$ is Convergent by ①. Hence by Comparison test, it follows from ②, that

$\sum 2|x_i y_i|$ is Convergent,

and therefore, $\sum |x_i y_i|$ is Convergent.

Now, $|x_i y_i| = |x_i||y_i| = |x_i||\overline{y_i}| = |x_i \overline{y_i}|$.

Therefore, $\sum |x_i \overline{y_i}|$ is Convergent.

i.e., $\sum x_i \overline{y_i}$ is absolutely Convergent.

Hence the inner-Product is well-defined.

$(x, x) = \sum x_i \overline{x_i} = \sum |x_i|^2 \geq 0$,

$(x, x) = 0$ iff $\sum |x_i|^2 = 0$ iff each $x_i = 0$ iff $x = 0$.

$(x, y) = \sum x_i \overline{y_i} = \sum \overline{y_i \overline{x_i}} = \overline{\sum y_i \overline{x_i}} = \overline{(y, x)}$

$(\alpha x + \beta y, z) = \sum (\alpha x_i + \beta y_i) \overline{z_i} = \alpha \sum x_i \overline{z_i} + \beta \sum y_i \overline{z_i}$

$= \alpha (x, z) + \beta (y, z)$ for all

~~$\alpha, \beta \in K, x, y, z \in l_2$.~~

$\alpha, \beta \in K, x, y, z \in l_2$.

Thus l_2 is an inner Product Space.

The norm defined by the inner Product is given by $\|x\| = (x, x)^{1/2} = (\sum |x_i|^2)^{1/2}$ and the metric d defined by this norm is given by

$$d(x,y) = \|x-y\| = \left(\sum |x_i - y_i|^2\right)^{1/2}.$$

We now show that (l_2, d) is a Complete metric space which will imply that the inner Product space l_2 is a Hilbert Space.

Let $(x^{(m)}) = ((x_i^{(m)}))$ be a Cauchy Sequence in l_2. Then given $\epsilon > 0$, there exists $n_0 \in N$ such that

$$d(x^{(m)}, x^{(n)}) = \left(\sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^2\right)^{1/2} < \epsilon \text{ for } m,n \geq n_0 \quad \text{— (A)}$$

Thus, for each fixed i, given $\epsilon > 0$, there exists $n_0 \in N$ such that,

$$|x_i^{(m)} - x_i^{(n)}| < \epsilon \text{ for } m,n \geq n_0.$$

Then for each fixed i, the numerical sequence $(x_i^{(m)})$ is a Cauchy Sequence and therefore, Converges to Some number x_i.

Put $x = (x_i)$. From inequality (A) we have

$$\sum_{i=1}^{r} |x_i^{(m)} - x_i^{(n)}|^2 < \epsilon^2 \text{ for } m,n \geq n_0 \text{ and every } r \in N.$$

Letting $m \to \infty$, we get

$$\sum_{i=1}^{r} |x_i - x_i^{(n)}|^2 \leq \epsilon^2 \text{ for } n \geq n_0 \text{ and every } r \in N.$$

Hence letting $r \to \infty$, we get

$$\sum_{i=1}^{\infty} |x_i - x_i^{(n)}|^2 \leq \epsilon^2 \text{ for } n \geq n_0 \quad \text{— (B)}$$

Thus, $x - x^{(n)} \in l_2$ for $n \geq n_0$. Now since l_2 is a linear space and $x^{(n)} \in l_2$ it follows that $x \in l_2$. Moreover, from inequality (B) it follows that

$d(x, x^{(n)}) \leq \epsilon$ for $n \geq n_0$.

Therefore, $(x^{(n)})$ Converges to $x \in l_2$. Thus (l_2, d) is a Complete metric Space. Hence the inner Product space l_2 is a Hilbert Space.

QNo> Example:- Construct an inner Product Space which is not a Hilbert Space or Construct an inComplete inner Product Space.

Soln:- The linear space $P[0,1]$ of all real valued Polynomials on $[0,1]$ with inner Product given by

$$(f, g) = \int_0^1 f(t)\, g(t)\, dt \quad \text{for } f, g \in P[0,1],$$ is an inner

Product Space which is not a Hilbert Space.

Clearly $(f, f) = \int_0^1 (f(t))^2\, dt \geq 0$,

$$(f, f) = 0 \text{ iff } \int_0^1 (f(t))^2\, dt = 0 \text{ iff } f(t) = 0$$

for every $t \in [0,1]$ and this holds iff $f = 0$.

$$(g, f) = \int_0^1 g(t) f(t)\, dt = \int_0^1 f(t)\, g(t)\, dt = (f, g).$$

$$(\alpha f + \beta g, h) = \int_0^1 (\alpha f(t) + \beta g(t))\, h(t)$$

$$= \alpha \int_0^1 f(t)\, h(t)\, dt + \beta \int_0^1 g(t)\, h(t)\, dt.$$

$$= \alpha (f, h) + \beta (g, h).$$

for all $\alpha, \beta \in \mathbb{R}$, $f, g, h \in P[0,1]$.

Therefore, the linear space $P[0,1]$ is an inner Product Space.

The norm defined by the ~~norm~~ inner Product is given by,

$$\|f\| = (f,f)^{1/2} = \left(\int_0^1 |f(t)|^2 dt\right)^{1/2}.$$

and the metric d defined by the norm is given by

$$d(f,g) = \|f-g\| = \left(\int_0^1 |f(t)-g(t)|^2 dt\right)^{1/2}.$$

The inner Product space $P[0,1]$ is not Complete. To See this, let $P_n(x) = \sum_{j=0}^{n} \frac{1}{2^j} x^j$. Then for

$$g(x) = \frac{1}{1-\frac{1}{2}x}, \quad 0 \leq x \leq 1, \quad P_n \to g \text{ in } L_2[0,1].$$

Thus, (P_n) is a Cauchy sequence in P which does not Converges to a vector in P since $g \notin P \subset L_2[0,1]$.

Thus, $P[0,1]$ is an inner Product space which is not a Hilbert space.

M.U. 96, ~~M.U.~~ M.Sc. 93, Problem:-

Q.No→ Show that every inner Product Space Can be embedded in a Hilbert space.

Soln:- Let E be a normed linear space. ~~Then there exists a Banach space E*~~ ~~Then there exi~~ We Show that every normed linear space E Can be embedded in a Banach Space E^* in the sense that E is isometrically isomorphic to a dense linear Subspace E_0 of E^*. If $x^*, y^* \in E^*$ and $(x_n) \in x^*$, $(y_n) \in y^*$, define an inner Product on E^* as follows

$(x^*, y^*) = \lim_{n\to\infty} (x_n, y_n)$. If $\| \ \|_1$ denotes the norm defined by the inner Product on E^* then

$$\|x^*\|_i = (x^*, x^*)^{1/2} = \lim_{n\to\infty} (x_n, x_n)^{1/2} = \lim_{n\to\infty} \|x_n\| = \|x^*\|$$

Thus, $\| \ \|_i$ is the Same as the original norm on E^*.

Hence E^* is a Hilbert space Containing E_0 as a dense linear subspace. Thus any inner product space can be embedded in a Hilbert space.

Q.No → Construct a Banach space of Continuous functions which is not a Hilbert space.

Solution:- Consider the Banach space $C[0,1]$ of all Continuous functions on the Closed interval $[0,1]$ of $\mathbb{R}$ with the norm defined by,

$$\|f\| = \sup_{t\in[0,1]} |f(t)| \text{ for } f \in C[0,1];$$

This norm does not Satisfy the Parallelogram law. This can be seen by taking $f(t) = t$ and $g(t) = 1-t$ for $t \in [0,1]$. Then $\|f\| = 1$

$$= \|g\| = \|f+g\| = \|f-g\|.$$

Thus, $\|f+g\|^2 + \|f-g\|^2 = 1^2 + 1^2 = 2$.

While, $2[\|f\|^2 + \|g\|^2] = 2[1^2 + 1^2) = 2\times 2 = 4$.

Hence the Parallelogram law is not Satisfied for the sup. norm on $C[0,1]$. But in a Hilbert space the Parallelogram law must be Satisfied. Therefore, the Banach space $C[0,1]$ with sup. norm is a Banach space but not a Hilbert space.